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EREMENKO POINTS AND THE STRUCTURE OF THE ESCAPING SET

P. J. RIPPON AND G. M. STALLARD

ABSTRACT. Much recent work on the iterates of a transcendental entire function f has been motivated by Eremenko's conjecture that all the components of the escaping set $I(f)$ are unbounded. We prove several general results about the topological structure of $I(f)$ including the fact that if $I(f)$ is disconnected, then it contains uncountably many pairwise disjoint unbounded continua, all of which are subsets of $A_R(f)$, the 'core' of the fast escaping set. We also show that, for some $R > 0$, the set $A_R(f)$ is connected and has the structure of an infinite spider's web or it contains uncountably many unbounded connected F_σ sets. There are analogous results for the intersections of these sets with the Julia set when multiply connected wandering domains are not present, but very different results when such wandering domains are present. In proving these, we obtain the unexpected result that some types of multiply connected wandering domains have complementary components with no interior, indeed uncountably many.

1. INTRODUCTION

Let f be a transcendental entire function and denote by f^n , $n = 0, 1, 2, \dots$, the n th iterate of f . The *Fatou set* $F(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n \in \mathbb{N}}$ forms a normal family in some neighborhood of z , and the *Julia set* of f is the complement of $F(f)$. The components of $F(f)$ are called *Fatou components*. An introduction to the properties of these sets can be found in [5].

The *escaping set*

$$I(f) = \{z : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

was first studied in detail by Eremenko [12], who made what is known as 'Eremenko's conjecture', which states that all the components of $I(f)$ are unbounded. This conjecture remains unsolved, though there are several important classes of entire functions for which it is known to be true, including many functions for which $I(f)$ is connected (see [17], [25], [27], [28], [34] and [37], for example).

In general, the topological structure of $I(f)$ can be highly complicated (see [30, Theorems 1.2 and 1.8], for example) and until now the only general results about $I(f)$ which hold for *all* transcendental entire functions were the following:

- (a) $I(f) \cap J(f) \neq \emptyset$, $J(f) = \partial I(f)$ and $\overline{I(f)}$ has no bounded components;
- (b) $I(f)$ has at least one unbounded component and, more precisely, either $I(f)$ is connected or it has infinitely many unbounded components;
- (c) $I(f) \cup \{\infty\}$ is connected.

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The properties in (a) were obtained by Eremenko in [12], and we obtained properties (b) and (c) in [34, Theorem 1.1], [33, Theorem 4.1] and [36, Theorem 1.3] by using properties of the fast escaping set, $A(f)$, to be defined shortly.

In studying $I(f)$, two main topological structures have been identified, which occur widely. First, for many transcendental entire functions the set $I(f)$ contains a *Cantor bouquet*, and in some cases is a subset of a Cantor bouquet. This topological concept was introduced in [11] and then described in a more general form in [3], where it is defined to be a set that is ambiently homeomorphic to a straight brush. In particular, a Cantor bouquet consists of uncountably many unbounded curves. Amongst the transcendental entire functions that have this property are entire functions of finite order in the Eremenko–Lyubich class \mathcal{B} ; see [37].

The second topological structure of $I(f)$ which occurs for many transcendental entire functions is a *spider's web*. In [34], we defined a set S to be an (*infinite*) *spider's web* if S is connected and there exists a sequence (G_n) of bounded simply connected domains such that

$$G_n \subset G_{n+1} \text{ and } \partial G_n \subset S, \text{ for } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n=1}^{\infty} G_n = \mathbb{C}.$$

Amongst the transcendental entire functions for which $I(f)$ is a spider's web are many entire functions of order less than $1/2$, and all transcendental entire functions that have multiply connected Fatou components; see [33].

Our first result shows that in some sense the escaping set of *any* transcendental entire function contains at least one of these two structures. We recall from [34] that if $I(f)$ contains a spider's web, then it is a spider's web.

Theorem 1.1. *Let f be a transcendental entire function and suppose that $I(f)$ is not a spider's web. Then $I(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ sets.*

Remarks 1. There are examples of transcendental entire functions for which $I(f)$ is a spider's web and at the same time $I(f)$ contains uncountably many pairwise disjoint curves; for example, the Fatou function $f(z) = z + 1 + e^{-z}$ has an $I(f)$ spider's web and its escaping set contains uncountably many pairwise disjoint curves that form a Cantor bouquet; see [13].

2. The exponential function $f(z) = e^z$ is an example of a function for which $I(f)$ is *not* a spider's web [23]. So the conclusions of Theorem 1.1 hold for this function, even though in this case $I(f)$ is connected [28].

We can obtain a stronger conclusion if we strengthen the hypothesis about $I(f)$ to assume that its complement contains an unbounded continuum (that is, an unbounded closed connected set), in particular if we assume that $I(f)$ is disconnected. This requires a different method of proof to that of Theorem 1.1.

Theorem 1.2. *Let f be a transcendental entire function and suppose that there exists an unbounded continuum in $I(f)^c$.*

If D is any open disc meeting $J(f)$, then the set $I(f) \setminus D$ has uncountably many unbounded components that meet ∂D , each containing an unbounded continuum, and these components are separated in $\mathbb{C} \setminus D$ by unbounded continua in $I(f)^c$.

Remarks 1. Whenever $I(f)$ is disconnected, there exists a closed connected set $\Gamma \subset I(f)^c$ such that $I(f)$ meets at least two complementary components of Γ ; see [27]. The set Γ can have no bounded complementary components, by [33, Theorem 4.1], so it must be an unbounded continuum. Thus $I(f)^c$ contains an unbounded continuum whenever $I(f)$ is disconnected.

2. In [22], we defined a *weak spider's web* to be a set with no unbounded continuum in its complement. Thus the hypothesis of Theorem 1.2 is that $I(f)$ is *not* a weak spider's web.

3. As noted above, if $I(f)$ is disconnected, then it must have infinitely many unbounded components. It is not known whether in this case $I(f)$ must have *uncountably* many components. Theorem 1.2 shows that if $I(f)$ is disconnected and has only countably many components, then the topological structure of $I(f)$ must be extremely complicated.

4. There are versions of Theorems 1.1 and 1.2 in which $I(f)$ is replaced throughout by a subset of $I(f)$ called the fast escaping set $A(f)$, defined below. We omit the statements of these $A(f)$ versions, whose proofs are similar to those for the $I(f)$ versions.

We prove Theorems 1.1 and 1.2 by using the core $A_R(f)$ of the *fast escaping set* $A(f)$, introduced in [8], which can be defined as follows; see [34]. Put

$$(1.1) \quad A_R(f) = \{z : |f^n(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\},$$

where $M(r) = M(r, f) = \max\{|f(z)| : |z| = r\}$, $r > 0$, $M^n(r) = M^n(r, f)$ denotes the n -th iterate of $r \mapsto M(r, f)$, and $R > 0$ is so large that $M(r) > r$ for $r \geq R$, and then put

$$A(f) = \{z : \text{for some } \ell \in \mathbb{N}, f^\ell(z) \in A_R(f)\}.$$

This definition of $A(f)$ is independent of R .

The set $A(f)$ has stronger properties than $I(f)$; for example, the components of $A_R(f)$ and $A(f)$ are always unbounded [33]. Also, if the set $A_R(f)$ is a spider's web, for some $R > 0$, then so are $A(f)$ and $I(f)$ [34] (in particular, these sets are connected), and moreover $I(f)$ has many strong properties [21].

We deduce Theorem 1.1 from a general result about $A_R(f)$. To state this result, we define

$$(1.2) \quad R(f) = \inf\{R \in [0, \infty) : M(r) > r, \text{ for } r \geq R\},$$

which is the least number such that $A_R(f)$ can be defined for all $R > R(f)$. We recall that if $A_R(f)$ is a spider's web for some $R > R(f)$, then $A_R(f)$ is a spider's web for *all* $R > R(f)$; see [34, Lemma 7.1(d)].

There are many classes of entire functions for which $A_R(f)$ is a spider's web and there are also many classes for which $A_R(f)$ contains uncountably many pairwise disjoint unbounded connected sets (indeed in many cases uncountably many unbounded curves); see [34, Section 8] and [29]. The following theorem shows that for *any* entire function one of these two extreme situations must occur for many values of R .

Theorem 1.3. *Let f be a transcendental entire function and let $R(f)$ be given by (1.2). Then one of the following holds.*

- (a) $A_R(f)$ is a spider's web for all $R > R(f)$.

- (b) *There is a dense set of values of $R \in (R(f), \infty)$ for which $A_R(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ sets.*

Theorem 1.1 is an immediate consequence of Theorem 1.3; indeed, if $I(f)$ is not a spider's web, then $A_R(f)$ is not a spider's web for any value of R , so $I(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ subsets of $A_R(f)$ for many values of $R > 0$.

Theorem 1.3 is proved using several different ideas. We begin by significantly refining Eremenko's original construction [12] of points in $I(f)$, drawing out the implications of the classical Wiman–Valiron theory on which his method is based, in a way that may have wider applications. This enables us to construct uncountably many points in $A_R(f)$ for a particular value of R , each with a distinct type of itinerary, which we call a *Wiman–Valiron itinerary*. Each of these *Eremenko points* lies in a certain unbounded connected subset of $A_R(f)$, and we then use delicate arguments involving conformal mapping and the theory of prime ends to show that if two of these unbounded connected subsets of $A_R(f)$ corresponding to distinct Eremenko points coincide, then $A_R(f)$ is a spider's web. The fact that $A_R(f)$ is closed is essential to our arguments.

The proof of Theorem 1.2 is rather different. It again builds on our construction of Eremenko points, but uses the method of their construction to pull back unbounded continua in $A(f)$ and $I(f)^c$ in a systematic manner.

Next, we discuss results relating to the intersections of $I(f)$, $A(f)$ and $A_R(f)$ with the Julia set of f . We recall that if all the Fatou components of f are simply connected, that is, f has *no* multiply connected wandering domains, then all the components of $J(f)$, and also of $A_R(f) \cap J(f)$ are unbounded; see [34, Theorem 1.3]. Moreover, in this situation all the Eremenko points of f lie in $J(f)$; see [12] and Theorem 3.3 (c) below. Therefore, for the intersections with $J(f)$, our proofs give analogous results to Theorems 1.1, 1.2 and 1.3, such as the following, whose proof we omit.

Theorem 1.4. *Let f be a transcendental entire function with no multiply connected wandering domains and let $R(f)$ be given by (1.2). Then one of the following holds.*

- (a) $A_R(f) \cap J(f)$ is a spider's web for all $R > R(f)$.
 (b) *There is a dense set of values of $R \in (R(f), \infty)$ for which $A_R(f) \cap J(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ sets.*

Finally, we consider the situation where f *does* have a multiply connected wandering domain. In this case the sets $I(f)$, $A(f)$ and $A_R(f)$ are all connected, and are in fact spiders' webs [34, Theorem 1.5], whereas the components of $I(f) \cap J(f)$, $A(f) \cap J(f)$ and $A_R(f) \cap J(f)$ are all bounded. We first show that in this case there are *always* uncountably many components of $I(f) \cap J(f)$.

Theorem 1.5. *Let f be a transcendental entire function with a multiply connected wandering domain. Then $I(f) \cap J(f)$ has uncountably many components, each of which is bounded.*

The proof of Theorem 1.5 shows that within each component of $I(f) \cap J(f)$ points all escape at the same rate, and that this rate of escape is extremely variable across the range of such components.

In contrast, we show that the sets $A(f) \cap J(f)$ and $A_R(f) \cap J(f)$ may have either countably many or uncountably many components in the case of multiply connected wandering domains. The term ‘inner connectivity’ used here is explained and discussed in detail in Section 7.

Theorem 1.6. (a) *Let f be a transcendental entire function with a multiply connected wandering domain U and suppose that $R > R(f)$. If U has infinite inner connectivity, then U has uncountably many complementary components, and hence $A_R(f) \cap J(f)$ and $A(f) \cap J(f)$ have uncountably many components, each of which is bounded.*

(b) *There exists a transcendental entire function f with a multiply connected wandering domain and $R > R(f)$ such that $A_R(f) \cap J(f)$ and $A(f) \cap J(f)$ have only countably many components, each of which is bounded.*

Remarks 1. Theorem 1.6 (a) shows that there exist transcendental entire functions with multiply connected wandering domains that have uncountably many complementary components with no interior. See Theorem 8.1 for more details.

2. The function in Theorem 1.6 (b) is a remarkable example due to Chris Bishop [10], constructed to solve a longstanding open problem of whether there is a transcendental entire function with Julia set of Hausdorff dimension 1, and we show here that this function has yet another surprising property.

The plan of the paper is as follows. Section 2 contains some topological results we use in several proofs. Section 3 contains our construction of uncountably many Eremenko points. Further properties of this Eremenko points construction are then given in Section 4, followed by the proof of Theorem 1.3 in Section 5. The proof of Theorem 1.2 is then given in Section 6. Section 7 contains background material on multiply connected wandering domains, Section 8 gives our new results on the structure of such wandering domains, and Section 9 contains the proofs of Theorems 1.5 and 1.6. Note that Sections 7, 8 and 9 can be read independently of the earlier sections. Finally, in Section 10 we state some open problems related to our results.

2. TOPOLOGICAL PRELIMINARIES

To prove our theorems, we need several results from point set topology, which we state here for the reader’s convenience. The two results in the following lemma are classical; see [19, pages 84 and 143].

Lemma 2.1. (a) *If E_0 is a continuum in $\hat{\mathbb{C}}$, E_1 is a closed subset of E_0 and C is a component of $E_0 \setminus E_1$, then \overline{C} meets E_1 .*
 (b) *If C_1 and C_2 are two components of a closed set E in $\hat{\mathbb{C}}$, then there is a Jordan curve in $\hat{\mathbb{C}} \setminus E$ that separates C_1 and C_2 .*

Lemma 2.1 (a) has two corollaries that we use frequently.

Corollary 2.2. *Let Γ be an unbounded continuum which meets the circle $C = \{z : |z| = r\}$, where $r > 0$. Then $\Gamma \cap \{z : |z| \geq r\}$ has at least one component that is an unbounded continuum, Γ' say, and any such component satisfies $\Gamma' \cap C \neq \emptyset$.*

Proof. The set $\hat{\Gamma} = \Gamma \cup \{\infty\}$ is a continuum in $\hat{\mathbb{C}}$. Then, by Lemma 2.1 (a), the closure of every component of $\hat{\Gamma} \setminus \{z : |z| \leq r\}$ meets C . Let $\hat{\Gamma}'$ be the closure of the component of $\hat{\Gamma} \setminus \{z : |z| \leq r\}$ that contains ∞ . Then any component, Γ' say, of $\hat{\Gamma}' \setminus \{\infty\}$ that meets C is unbounded, by Lemma 2.1 (a) again. In fact, every component of $\hat{\Gamma}' \setminus \{\infty\}$ must meet C , since otherwise we could obtain a contradiction to the connectedness of $\Gamma \cup \{z : |z| \leq r\}$. \square

Our next corollary involves certain unbounded connected subsets of an unbounded continuum, which play a key role in this paper.

Corollary 2.3. *Let Γ be an unbounded continuum and suppose that $z \in \Gamma$. Then*

$$(2.1) \quad \Gamma(z) = \bigcup \{K : K \text{ is a continuum, } z \in K, K \subset \Gamma\}$$

is an unbounded connected F_σ subset of Γ . Also, if $\Gamma(z) \cap \Gamma(z') \neq \emptyset$, where $z, z' \in \Gamma$, then $\Gamma(z) = \Gamma(z')$.

Proof. It follows from Lemma 2.1 (a) that for all $r > |z|$ the set $\Gamma \cap \{z : |z| \leq r\}$ contains a unique continuum, $\Gamma_r(z)$ say, which contains z and also meets the circle $\{z : |z| = r\}$. Clearly $\Gamma_r(z) \subset \Gamma(z)$ for all $r > |z|$, so $\Gamma(z)$ is nonempty and unbounded, and evidently connected. To see that $\Gamma(z)$ is an F_σ set, note that for any continuum K such that $z \in K$ and $K \subset \Gamma \cap \{z : |z| \leq r\}$ we have $K \subset \Gamma_r(z)$, so $\Gamma_{r_1}(z) \subset \Gamma_{r_2}(z)$, for $r_2 > r_1 > |z|$, and

$$\Gamma(z) = \bigcup_{r > |z|} \Gamma_r(z) = \bigcup_{n \in \mathbb{N}, n > |z|} \Gamma_n(z). \quad \square$$

Remark In many cases the set $\Gamma(z)$ defined in (2.1) is itself an unbounded continuum, but not always.

Finally we shall need the following simple topological lemma; see [32, Lemma 1].

Lemma 2.4. *Let E_n , $n \geq 0$, be a sequence of non-empty compact sets in \mathbb{C} and $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ be a continuous function such that*

$$(2.2) \quad f(E_n) \supset E_{n+1}, \quad \text{for } n \geq 0.$$

Then there exists ζ such that $f^n(\zeta) \in E_n$, for $n \geq 0$.

If f is also meromorphic and $E_n \cap J(f) \neq \emptyset$, for $n \geq 0$, then there exists $\zeta \in J(f)$ such that $f^n(\zeta) \in E_n$, for $n \geq 0$.

3. CONSTRUCTING UNCOUNTABLY MANY EREMENKO POINTS

It was shown in [8] that Eremenko's construction in [12] of points in $I(f)$ actually gives points that are in $A(f)$. Points constructed in this way have particularly nice properties and as noted earlier we often refer to them as *Eremenko points*; see [31] and [7]. Eremenko's construction was based on Wiman–Valiron theory, and here we use a modification of this construction to give uncountably many such points. In Theorem 3.1 we give a key result of Wiman–Valiron theory (see

[16], for example) which describes the behaviour of an entire function f near points at which f takes its maximum modulus.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire function and, for $r > 0$, let $z(r)$ denote a point such that

$$|z(r)| = r \text{ and } |f(z(r))| = M(r).$$

Also, let $N(r)$ be the largest value of n for which $|a_n| r^n$ is maximal. Note that $N(r)$ is increasing with r and $N(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Theorem 3.1. *Let f be a transcendental entire function and let $\alpha > 1/2$. There exists a set $E \subset (0, \infty)$ such that $\int_E (1/t) dt < \infty$ and, for $r \in (0, \infty) \setminus E$,*

$$(3.1) \quad f(z) = \left(\frac{z}{z(r)} \right)^{N(r)} f(z(r)) (1 + \varepsilon(r, z, \alpha)), \text{ for } |z - z(r)| < r/(N(r))^\alpha,$$

where $\varepsilon(r, z, \alpha) \rightarrow 0$ uniformly with respect to z as $r \rightarrow \infty, r \notin E$.

We need the following consequence of Theorem 3.1, which is related to [34, Theorem 2.4] but gives more precise information.

Theorem 3.2. *Let f be a transcendental entire function, $K \geq 20\pi$, and put*

$$(3.2) \quad D_{r,K} = \{z : |z - z(r)| < Kr/N(r)\}, \quad r > 0.$$

Then there exists a set $E_K \subset (0, \infty)$ with $\int_{E_K} (1/t) dt < \infty$ such that, if $r \in [1, \infty) \setminus E_K$, then

- *the disc $D_{r,K}$ contains a closed quadrilateral $Q_{r,K}$ that can be partitioned into quadrilaterals $Q_{r,K,j}$, where $j \in \mathbb{Z}$, $|j| \leq K/(10\pi)$, labelled in anti-clockwise order with respect to the origin, such that $z(r) \in Q_{r,K,0}$ and the interior of each $Q_{r,K,j}$ is a univalent preimage under f of the cut annulus*

$$\{w : \tfrac{1}{2}M(r) < |w| < 2M(r), |\arg(w/f(z(r)))| < \pi\};$$

- *if B is a compact subset of any disc D in the cut annulus above and B_{-1} is a component of $f^{-1}(B) \cap Q_{r,K}$, then*

$$(3.3) \quad \text{diam } B \geq c(f, K) N(r) \frac{M(r)}{r} \text{diam } B_{-1},$$

where $c = c(f, K) > 0$ is a constant that depends only on f and K .

Proof. Let $\alpha = 3/4$ and let E be the corresponding exceptional set defined in Theorem 3.1. Since $\alpha < 1$, it follows from Theorem 3.1 together with the fact that $N(r) \rightarrow \infty$ as $r \rightarrow \infty$ that we can take $r(f, K) > 0$ so large that, for $r \geq r(f, K)$, $r \notin E$, and $z \in D_{r,K}$, we have the linear approximation

$$(3.4) \quad \begin{aligned} \log \left(\frac{f(z)}{f(z(r))} \right) &= N(r) \log \left(\frac{z}{z(r)} \right) + \log(1 + \varepsilon(r, z)) \\ &= N(r) \left(\frac{z - z(r)}{z(r)} \right) + \varepsilon_1(r, z), \end{aligned}$$

where $|\varepsilon_1(r, z)| \leq 1/100$. We can now use Rouché's theorem to deduce from (3.4) that we can also take $r(f, K) > 0$ so large that if $r \geq r(f, K)$, $r \notin E$, then the function

$$g(z) = \log(f(z)/f(z(r)))$$

maps $D_{r,K}$ univalently, with $g(z(r)) = 0$, and that $g(D_{r,K})$ contains the disc with centre 0 and radius $\frac{1}{2}K$, and hence contains the square

$$\{w : |\Re w| \leq \frac{1}{4}K, |\Im w| \leq \frac{1}{4}K\}.$$

Now, since $K \geq 20\pi$, this square contains all rectangles of the form

$$R_j = \{w : |\Re w| < \log 2, |\Im w - 2j\pi| < \pi\}, \quad j \in \mathbb{Z}, |j| \leq K/(10\pi).$$

Since $K/(10\pi) \geq 2$ and $|f(z(r))| = M(r)$, we deduce that, for $r \geq r(f, K)$, $r \notin E$, there are at least five quadrilaterals of the form $Q_{r,K,j} = g^{-1}(R_j)$ in $D_{r,K}$ such that $z(r) \in Q_{r,K,0}$, f acts univalently on $Q_{r,K,j}$, and

$$\begin{aligned} f(Q_{r,K,j}) &= f(z(r)) \exp(R_j) \\ &= \{w : \tfrac{1}{2}M(r) < |w| < 2M(r), |\arg(w/f(z(r)))| < \pi\}, \end{aligned}$$

as required. The union of the closures of these $Q_{r,K,j}$, $|j| \leq K/(10\pi)$, forms the required quadrilateral $Q_{r,K}$, which contains $z(r)$ since $z(r) \in Q_{r,K,0}$.

The estimate (3.3) follows by expressing f in $D_{r,K}$ as in (3.4) and again considering the function in two stages as $f(z) = f(z(r)) \exp(g(z))$. \square

Next, we use Theorem 3.2 to construct uncountably many Eremenko points, each with a different ‘Wiman–Valiron itinerary’ determined by a sequence of quadrilaterals to which Theorem 3.2 has been applied successively, and each lying in $A_R(f)$ for the same suitably defined value of R . In the proof of Theorem 3.3 we use the following notation for an open annulus:

$$A(r, R) = \{z : r < |z| < R\}, \quad 0 < r < R.$$

Theorem 3.3. *Let f be a transcendental entire function and let $R(f)$ be defined by (1.2). There exists $R_1(f) \geq R(f)$ such that if $r_0 \geq R_1(f)$, then there exist sequences of positive numbers (r_n) , complex numbers (z_n) , and quadrilaterals (Q_n) , each of which can be partitioned into the union of five quadrilaterals with interiors $Q_{n,j}$, for $j = -2, -1, 0, 1, 2$, labelled in anticlockwise order with respect to the origin, such that, for $n \geq 0$,*

$$(3.5) \quad \tfrac{5}{4}r_n \leq |z_n| \leq \tfrac{7}{4}r_n \text{ and } r_{n+1} = |f(z_n)| = M(|z_n|),$$

$$(3.6) \quad z_n \in Q_{n,0} \subset Q_n \subset A(r_n, 2r_n),$$

and

$$(3.7) \quad f \text{ maps } Q_{n,j} \text{ univalently onto } A_{n+1}, \text{ for } j = -2, -1, 0, 1, 2,$$

where

$$A_{n+1} = \{w : \tfrac{1}{2}r_{n+1} < |w| < 2r_{n+1}, |\arg(w/f(z_n))| < \pi\}.$$

Furthermore,

(a) *the sequence $M^{-n}(r_n)$, $n \geq 0$, is strictly increasing and its limit R satisfies*

$$(3.8) \quad r_0 < |z_0| < R < 2r_0;$$

(b) *for each sequence of the form $j_n = \pm 1$, $n \geq 0$, there exists a unique point $z(j_n) \in \overline{Q_{0,j_0}}$ with Wiman–Valiron itinerary $(j_n)_{n \geq 0}$, in the sense that*

$$(3.9) \quad f^n(z(j_n)) \in \overline{Q_{n,j_n}}, \quad \text{for } n \geq 0,$$

and $z(j_n) \in A_R(f)$;

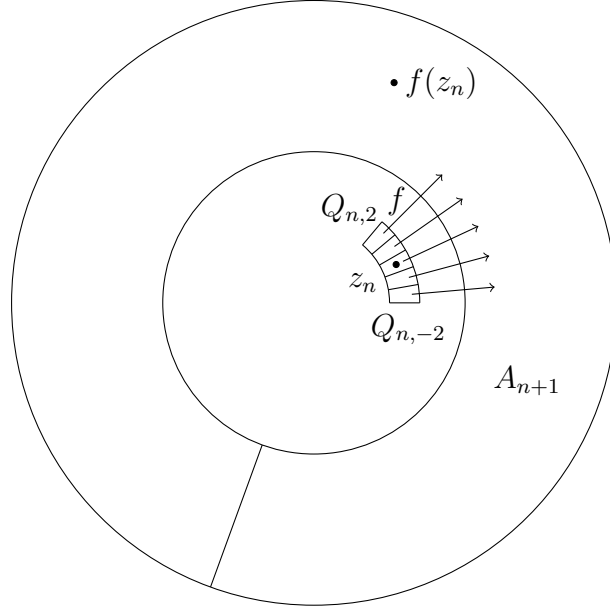


FIGURE 1. The quadrilaterals $Q_{n,j}$, $j = -2, \dots, 2$, and the cut annulus A_{n+1}

- (c) if f has no multiply connected wandering domains, then each $z(j_n)$ lies in $J(f)$;
- (d) for each sequence of the form $j_n = \pm 1$, $n \geq 0$, and $k \in \mathbb{N}$, $f^k(z(j_n))$ is the unique point in $\overline{Q_{k,j_k}}$ with itinerary $(j_{k+n})_{n \geq 0}$, and $f^k(z(j_n)) \in A_{M^k(R)}(f)$.

The quadrilaterals $Q_{n,j}$, $j = -2, \dots, 2$, and the cut annulus A_{n+1} are illustrated schematically in Figure 1.

Remarks 1. It will be clear from the proof of Theorem 3.3 that the constructed Eremenko point $z(j_n)$ actually satisfies $f^n(z(j_n)) \in Q_{n,j_n}$, for $n \geq 0$.

2. In this proof, and in later proofs in this paper, we often use the fact that

$$(3.10) \quad z \in A_R(f) \text{ if and only if } f^k(z) \in A_{M^k(R)}(f), \text{ for } R > R(f), k \in \mathbb{N},$$

which follows immediately from the definition of $A_R(f)$ in (1.1).

In the proof of Theorem 3.3 and later in the paper we need the following basic properties of the maximum modulus of a transcendental entire function; see [31, Lemma 2.2].

Lemma 3.4. *Let f be a transcendental entire function. There exists $R_0(f) > 0$ such that*

$$(3.11) \quad M(r^c) \geq M(r)^c, \text{ for } r \geq R_0(f), c > 1,$$

and it follows that for any $k > 1$ we have

$$(3.12) \quad M(kr)/M(r) \rightarrow \infty \text{ as } r \rightarrow \infty.$$

Proof of Theorem 3.3. We apply Theorem 3.2 with $K = 20\pi$ and E_K the corresponding exceptional set given by Theorem 3.2. Then there exists $R_1(f) > 1$ so large that

$$(3.13) \quad \int_{E_K \cap (R_1(f), \infty)} \frac{1}{t} dt < \log \frac{7}{5},$$

and

$$(3.14) \quad \frac{K}{N(r)} < \frac{1}{8} \text{ and } M(r) > r, \text{ for } r \geq R_1(f).$$

By (3.12), we can also choose $R_1(f)$ so large that

$$(3.15) \quad M\left(\frac{9}{8}r\right) \geq 2M(r), \text{ for } r \geq R_1(f).$$

As in Theorem 3.2, we let $D_{r,K} = \{z : |z - z(r)| < Kr/N(r)\}$ and note that, by the first statement in (3.14),

$$(3.16) \quad D_{r,K} \subset A\left(\frac{7}{8}r, \frac{9}{8}r\right), \text{ for } r \geq R_1(f).$$

Take $r_0 \geq R_1(f)$. It follows from Theorem 3.2 and (3.13) that there exists

$$(3.17) \quad r'_0 \in \left[\frac{5}{4}r_0, \frac{7}{4}r_0\right] \setminus E_K,$$

and $z_0 = z(r'_0)$ such that

- the disc $D_0 = D_{r'_0,K}$ contains a quadrilateral Q_0 that can be partitioned into five quadrilaterals $Q_{0,j}$, $j = -2, -1, 0, 1, 2$, with $z_0 \in Q_{0,0}$, the interior of each of which is a univalent preimage under f of the cut annulus

$$A_1 = \{w : \frac{1}{2}M(|z_0|) < |w| < 2M(|z_0|), |\arg(w/f(z_0))| < \pi\};$$

- if B is a compact subset of any disc $D \subset A_1$ and B_{-1} is a component of $f^{-1}(B) \cap Q_0$, then

$$\text{diam } B \geq c(f, K)N(|z_0|) \frac{M(|z_0|)}{|z_0|} \text{diam } B_{-1},$$

where $c = c(f, K) > 0$ is a constant that depends only on f and K .

Note that, by (3.16) and (3.17), we have

$$(3.18) \quad Q_0 \subset D_0 \subset A(r_0, 2r_0).$$

Repeating this process with $r_1 = |f(z_0)| = M(|z_0|)$ instead of r_0 , we deduce that there exists $r'_1 \in \left[\frac{5}{4}r_1, \frac{7}{4}r_1\right] \setminus E_K$ and $z_1 = z(r'_1)$ such that

- the disc $D_1 = D_{r'_1,K}$ contains a quadrilateral Q_1 that can be partitioned into five quadrilaterals $Q_{1,j}$, $j = -2, -1, 0, 1, 2$, with $z_1 \in Q_{1,0}$, the interior of each of which is a univalent preimage under f of the cut annulus

$$A_2 = \{w : \frac{1}{2}M(|z_1|) < |w| < 2M(|z_1|), |\arg(w/f(z_1))| < \pi\};$$

- if B is a compact subset of any disc $D \subset A_2$ and B_{-1} is a component of $f^{-1}(B) \cap Q_1$, then

$$\text{diam } B \geq c(f, K)N(|z_1|) \frac{M(|z_1|)}{|z_1|} \text{diam } B_{-1}.$$

Carrying out this process repeatedly, we obtain sequences of positive numbers (r_n) , complex numbers (z_n) such that $\frac{5}{4}r_n \leq |z_n| \leq \frac{7}{4}r_n$, and, for $n \geq 0$, discs

$$(3.19) \quad D_n = \{z : |z - z_n| < K|z_n|/N(|z_n|)\} \subset A\left(\frac{7}{8}|z_n|, \frac{9}{8}|z_n|\right) \subset A(r_n, 2r_n),$$

quadrilaterals

$$Q_{n,j} \subset Q_n \subset D_n, \quad j = -2, -1, 0, 1, 2,$$

and cut annuli

$$A_{n+1} = \{w : \frac{1}{2}r_{n+1} < |w| < 2r_{n+1}, |\arg(w/f(z_n))| < \pi\},$$

that satisfy (3.5), (3.6) and (3.7), and also:

if B is a compact subset of any disc $D \subset A_{n+1}$ and B_{-1} is a component of $f^{-1}(B) \cap Q_n$, then

$$(3.20) \quad \text{diam } B \geq c(f, K)N(|z_n|)\frac{M(|z_n|)}{|z_n|} \text{diam } B_{-1}.$$

To prove part (a) of Theorem 3.3 we note that, by the construction and (3.15), we have, for $n \geq 0$,

$$r_{n+1} = M(|z_n|) > M(r_n) \quad \text{and} \quad M(2r_n) > M(\frac{9}{8}|z_n|) \geq 2M(|z_n|) = 2r_{n+1}.$$

Hence,

$$(3.21) \quad r_0 < |z_0| = M^{-1}(r_1) < M^{-2}(r_2) < \cdots < M^{-2}(2r_2) < M^{-1}(2r_1) < 2r_0,$$

from which part (a) follows.

Now let (j_n) denote any sequence whose elements are ± 1 . It follows from (3.6) and (3.7) that

$$f(\overline{Q_{n,j_n}}) \supset \overline{A_{n+1}} \supset \overline{Q_{n+1,j_{n+1}}}, \quad \text{for } n \geq 0.$$

Thus, given (j_n) , we can construct a sequence of compact sets B_n such that $B_0 = \overline{Q_{0,j_0}}$ and, for $n \in \mathbb{N}$, B_n is a component of $f^{-n}(\overline{Q_{n,j_n}})$ with $B_n \subset B_{n-1}$. Then $\bigcap_{n=0}^{\infty} B_n$ is a nested intersection of compact sets and is therefore non-empty.

To prove part (b) we show that $\bigcap_{n=0}^{\infty} B_n$ consists of a single point. By (3.20), applied with $B = f^{k+1}(B_n)$ and $B_{-1} = f^k(B_n)$ for $k = 0, 1, \dots, n-1$, and also (3.5), (3.6) and (3.19), we deduce that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \text{diam } B_n &\leq \left(\prod_{k=0}^{n-1} \frac{|z_k|}{c(f, K)N(|z_k|)M(|z_k|)} \right) \text{diam } \overline{Q_{n,j_n}} \\ &\leq \left(\prod_{k=0}^{n-1} \frac{|z_k|}{c(f, K)N(|z_k|)M(|z_k|)} \right) \frac{2K|z_n|}{N(|z_n|)} \\ &\leq \frac{K2^{n+1}|z_0|}{c(f, K)^n N(|z_0|)N(|z_1|) \cdots N(|z_n|)} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

since $Q_{n,j_n} \subset D_n$, $\text{diam } D_n = 2K|z_n|/N(|z_n|)$, and $N(r) \rightarrow \infty$ as $r \rightarrow \infty$. Thus $\bigcap_{n=0}^{\infty} B_n$ consists of a single point.

For the given sequence (j_n) , we let $\bigcap_{n=0}^{\infty} B_n = \{z(j_n)\}$. Then, for each $n \in \mathbb{N}$, $f^n(z(j_n)) \in \overline{Q_{n,j_n}}$, so (3.9) holds.

We now show that $z(j_n) \in A_R(f)$. If not, there exists $R' < R$ and $N_1 \in \mathbb{N}$ such that

$$|f^n(z(j_n))| \leq M^n(R'), \quad \text{for } n \geq N_1.$$

But, by (3.21), there also exists $N_2 \in \mathbb{N}$ such that

$$M^{-n}(r_n) > R', \quad \text{for } n \geq N_2,$$

so

$$|f^n(z(j_n))| < r_n, \quad \text{for } n \geq \max\{N_1, N_2\},$$

a contradiction to (3.6) and (3.9). This proves part (b).

To prove part (c), we observe that if there exists an open disc D such that $z(j_n) \in D \subset F(f)$, then there exists $N_0 \in \mathbb{N}$ such that

$$B_n \subset D, \text{ for } n \geq N_0.$$

Since B_n is a component of $f^{-n}(\overline{Q_{n,j_n}})$, we deduce that

$$\overline{Q_{n,j_n}} = f^n(B_n) \subset f^n(D) \subset F(f), \text{ for } n \geq N_0,$$

so

$$\overline{A_{n+1}} = f^{n+1}(B_n) \subset F(f), \text{ for } n \geq N_0,$$

and hence $\overline{A_{n+1}}$ is contained in a multiply connected wandering domain for n sufficiently large.

Finally, to prove part (d) we follow the construction in part (a), but start from Q_{k,j_k} instead of Q_{0,j_0} , with the itinerary (j_{k+n}) instead of (j_n) , and use the facts that $f^k(z) \in A_{M^k(R)}(f)$ whenever $z \in A_R(f)$, and

$$\lim_{n \rightarrow \infty} M^{-n}(r_{k+n}) = \lim_{n \rightarrow \infty} M^k(M^{-n-k}(r_{k+n})) = M^k(R). \quad \square$$

4. FURTHER PROPERTIES OF THE EREMENKO POINTS CONSTRUCTION

Theorem 3.3 shows that in each interval of the form $(r_0, 2r_0)$, where $r_0 \geq R_1(f)$, we can choose R with the property that there are points in $A_R(f)$ each of whose orbits passes through a sequence of quadrilaterals $Q_{n,j_n} \subset Q_n$, $n \geq 0$, corresponding to one of the uncountably many Wiman–Valiron itineraries (j_n) , $j_n = \pm 1$. To prove Theorem 1.3 we shall require some further properties of these quadrilaterals Q_{n,j_n} , each of which is a univalent preimage under f of

$$A_{n+1} = \{w : \tfrac{1}{2}r_{n+1} < |w| < 2r_{n+1}, |\arg(w/f(z_n))| < \pi\}.$$

We label the ‘inner edges’ of Q_n and Q_{n,j_n} , which are mapped under f to $\{w : |w| = \tfrac{1}{2}r_{n+1}\}$, as α_n and α_{n,j_n} , respectively.

Theorem 4.1. *Let f be a transcendental entire function, let (r_n) , (z_n) , (Q_n) , (Q_{n,j_n}) , $z(j_n)$ and R be as in Theorem 3.3, and let the inner edges of Q_n and Q_{n,j_n} be as defined above. Then, for $n \geq 0$,*

$$(4.1) \quad r_n < |z_n| < M^n(R) \leq |f^n(z(j_n))| < 2r_n,$$

and

$$(4.2) \quad \alpha_n \subset \{z : |z| < |z_n|\} \subset \{z : |z| < M^n(R)\}.$$

Also, if G_n is the component of $Q_{n,j_n} \setminus A_{M^n(R)}(f)$ whose boundary contains α_{n,j_n} , then G_n is simply connected,

$$(4.3) \quad Q_{n,j_n} \cap \{z : |z| < M^n(R)\} \subset G_n \text{ and } f^n(z(j_n)) \in \partial G_n.$$

Proof. The inequalities in (4.1) follow from the construction in Theorem 3.3 in the same way that those in part (a) of Theorem 3.3 do (this is the special case when $n = 0$), together with the fact that $z(j_n) \in A_R(f)$; see Theorem 3.3 (b).

Next, we prove property (4.2). It follows by (3.4) that, for $n \geq 0$, we have

$$(4.4) \quad g_n(z) = \log \left(\frac{f(z)}{f(z_n)} \right) = N(|z_n|) \left(\frac{z - z_n}{z_n} \right) + \varepsilon_1(z, |z_n|), \text{ for } z \in D_n,$$

where the disc D_n is defined by (3.19) and $|\varepsilon_1(z, |z_n|)| \leq 1/100$. The function g_n maps D_n univalently, with $g_n(z_n) = 0$, and $g_n(D_n)$ contains the square

$$\{w : |\Re w| \leq 5\pi, |\Im w| \leq 5\pi\},$$

since $K = 20\pi$ in Theorem 3.3. Also, for $n \geq 0$,

$$Q_n = g_n^{-1}(\{w : |\Re w| < \log 2, |\Im w| < 5\pi\}),$$

and

$$Q_{n,j} = g_n^{-1}(\{w : |\Re w| < \log 2, |\Im w - 2j\pi| < \pi\}), \quad j = -2, -1, 0, 1, 2.$$

Since α_n is the ‘inner’ edge of Q_n , we deduce by considering the inverse of the linear approximation (4.4) that

$$\alpha_n \subset \{z : |z| < |z_n|\} \subset \{z : |z| < M^n(R)\},$$

as required.

The domain G_n is simply connected because the set $A_{M^n(R)}(f)$ has no bounded components. The first part of (4.3) follows from the definition of G_n and the fact that $A_{M^n(R)}(f) \subset \{z : |z| \geq M^n(R)\}$.

To prove the second part of (4.3), note first that f maps G_n univalently onto a simply connected domain whose boundary is contained in $\partial A_{n+1} \cup A_{M^{n+1}(R)}(f)$, by (3.10). Since $Q_{n+1} \subset \{z : r_{n+1} < |z| < 2r_{n+1}\}$, we deduce that $f(G_n)$ contains the domain G_{n+1} defined as the component of $Q_{n+1,j_{n+1}} \setminus A_{M^{n+1}(R)}(f)$ whose boundary contains $\alpha_{n+1,j_{n+1}}$.

Repeating this process, we obtain a sequence of domains $G_m \subset Q_{m,j_m}$, $m \geq n$, such that

$$f(\overline{G_m}) \supset \overline{G_{m+1}}, \quad m \geq n.$$

Hence $\overline{G_n}$ contains a point z' such that

$$f^{m-n}(z') \in \overline{G_m} \subset \overline{Q_{m,j_m}}, \quad \text{for } m \geq n,$$

by Lemma 2.4. It now follows from the final statement of Theorem 3.3 that $z' = f^n(z(j_n))$, so $f^n(z(j_n)) \in \overline{G_n}$. Since $f^n(z(j_n)) \in A_{M^n(R)}(f)$ it follows that $f^n(z(j_n)) \in \partial G_n$. This completes the proof of Theorem 4.1. \square

5. PROOF OF THEOREM 1.3

In the proof of Theorem 1.3, we shall suppose that for every $R > R(f)$ the set $A_R(f)$ is *not* a spider’s web and deduce that there exists a dense set of values of $R \in (R(f), \infty)$ such that $A_R(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ sets. First, recall that for all $R > R(f)$, the set $A_R(f)$ has the property that each of its components is closed and unbounded, and lies in $\{z : |z| \geq R\}$; see [34, Theorem 1.1].

We continue to use all the notation from Theorems 3.3 and 4.1, and make use of the results proved there. Suppose that $r_0 \geq R_1(f)$ and let R be given by Theorem 3.3(a), so $r_0 < R < 2r_0$. Each of the uncountably many Eremenko points $z = z(j_n)$ found in Theorem 3.3 lies in an unbounded closed component of $A_R(f)$, say $\Gamma(j_n)$.

We now introduce $\Gamma(z(j_n))$, the unbounded connected F_σ subset of $\Gamma(j_n)$ containing $z(j_n)$, defined by (2.1), and show that, because $A_R(f)$ is not a spider’s web,

the sets $\Gamma(z(j_n))$ are pairwise disjoint. This proves the statement of Theorem 1.3 for this value of R .

Suppose then that $z(j_n)$ and $z(j'_n)$ are two Eremenko points, where (j_n) and (j'_n) are different Wiman–Valiron itineraries, but $\Gamma(z(j_n))$ and $\Gamma(z(j'_n))$ are not disjoint. Then, by Corollary 2.3, there exists a continuum K such that

$$(5.1) \quad z(j_n), z(j'_n) \in K \text{ and } K \subset A_R(f).$$

Since (j_n) and (j'_n) are distinct, we can take $N \geq 0$ such that $f^N(z(j_n))$ and $f^N(z(j'_n))$ lie in distinct quadrilaterals Q_{N,j_N} and Q_{N,j'_N} , respectively. We note that

$$f^N(z(j_n)), f^N(z(j'_n)) \in f^N(K) \text{ and } f^N(K) \subset A_{M^N(R)}(f),$$

by (5.1) and (3.10), and $f^N(K)$ is a continuum.

Without loss of generality we can assume that $N = 0$. We can also assume, by relabelling, that $z(j_n) \in Q_{0,-1} \subset Q_0$ and $z(j'_n) \in Q_{0,1} \subset Q_0$. For simplicity, we write $z = z(j_n)$, $z' = z(j'_n)$, $Q = Q_{0,-1}$ and $Q' = Q_{0,1}$, and denote the inner edges of Q and Q' by α and α' , respectively. We know from the proof of Theorem 3.3 that f maps both Q and Q' conformally onto a cut annulus of the form

$$A = \{w : \tfrac{1}{2}M(|z_0|) < |w| < 2M(|z_0|), |\arg(w/f(z_0))| < \pi\},$$

where $z_0 \in Q_{0,0}$, and f maps both α and α' onto the inner boundary component $\{w : |w| = \tfrac{1}{2}M(|z_0|)\}$ of \bar{A} ; see Figure 2.

Then let G denote the component of $Q \setminus A_R(f)$ whose boundary contains α and let G' denote the component of $Q' \setminus A_R(f)$ whose boundary contains α' . We have

$$\alpha \cup \alpha' \subset \{z : |z| < R\}, \quad Q \cap \{z : |z| < R\} \subset G \text{ and } Q' \cap \{z : |z| < R\} \subset G',$$

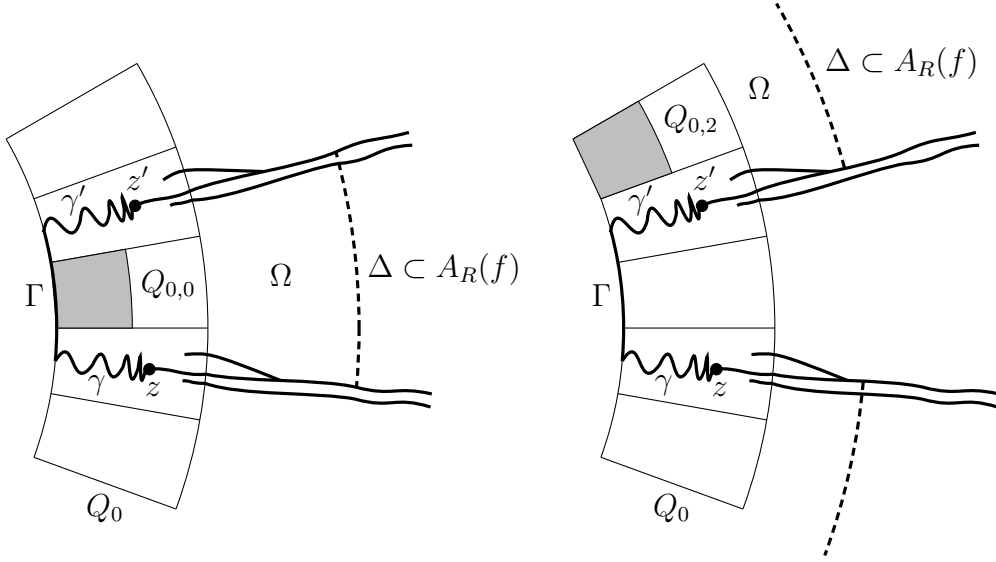
by (4.2) and (4.3) with $n = 0$. Both G and G' are simply connected, and are mapped by f conformally onto the component, H say, of $A \setminus A_{M(R)}(f)$ whose boundary contains $\{w : |w| = \tfrac{1}{2}r_1\}$, where $r_1 = M(|z_0|)$.

By the final statement of Theorem 4.1, in (4.3), we have $z \in \partial G$ and $z' \in \partial G'$. Now we take a simple path $\gamma \subset G$ for which $\bar{\gamma} \setminus \gamma$ is the union of a point, z_α say, that lies on the edge α and a continuum in ∂G that contains z but does not contain any open arcs of $\partial Q \setminus A_R(f)$. We can obtain such a path by, for example, using a Riemann mapping of G onto the open unit disc \mathbb{D} . Under this mapping, any prime end of G whose impression contains z corresponds to a point $\zeta_z \in \partial \mathbb{D}$ and z_α corresponds to a point $\zeta_{z_\alpha} \in \partial \mathbb{D}$. Then we can take γ to be the preimage in G of the path in \mathbb{D} consisting of the two radii from 0 to ζ_z and from 0 to ζ_{z_α} ; see [24] for the theory of prime ends and, in particular, Carathéodory's theorem giving the correspondence between prime ends and boundary points of the open unit disc.

Similarly, take a simple path $\gamma' \subset G'$ for which $\bar{\gamma'} \setminus \gamma'$ is the union of a point on α' and a continuum in $\partial G'$ that contains z' but does not contain any open arcs of $\partial Q' \setminus A_R(f)$. Then let Γ denote the union of the paths γ , γ' and the segment of the inner edge of Q_0 that joins the endpoints of these two paths in α and α' .

Now recall from (5.1) that z and z' both lie in a continuum $K \subset A_R(f)$ and put $\Delta = K \cup (\bar{\Gamma} \setminus \Gamma)$. Then Δ is a continuum, since $z, z' \in \bar{\Gamma} \setminus \Gamma$, and we note that

$$\bar{\Gamma} \setminus \Gamma \subset A_R(f), \quad \text{so} \quad \Delta \subset A_R(f).$$

FIGURE 2. Two possible locations of the continuum Δ and the domain Ω

The statement here that $\bar{\Gamma} \setminus \Gamma \subset A_R(f)$ is true because $A_R(f)$ is closed and Γ does not accumulate at any points of ∂G or $\partial G'$ that are outside $A_R(f)$.

We consider the bounded domain Ω which is a component of the complement of $\Delta \cup \Gamma = \bar{\Gamma} \cup K$ and whose boundary consists of Γ and a subset of the continuum Δ . Since $\Delta \subset A_R(f) \subset \{z : |z| \geq R\}$, we deduce by (4.2) and (4.3) that at least one of the following must be the case:

$$Q_{0,0} \cap \{z : |z| < |z_0|\} \subset \Omega \quad \text{or} \quad Q_{0,2} \cap \{z : |z| < |z_0|\} \subset \Omega,$$

depending on the location of the continuum Δ . The two possibilities are illustrated in Figure 2.

In either case, we deduce that $f(\Omega)$ is a bounded domain that contains an open annulus of the form $A(\frac{1}{2}r_1, \frac{1}{2}r_1 + \varepsilon)$, for some $\varepsilon > 0$.

Now,

$$\partial f(\Omega) \subset f(\partial\Omega) \subset f(\Gamma) \cup f(\Delta),$$

and we claim that $f(\Gamma)$ does not meet the boundary, C say, of the set which is the union of $f(\Omega)$ and its bounded complementary components. This is clearly the case for the part of Γ lying in the inner edge of Q_0 , which maps to $\{w : |w| = \frac{1}{2}r_1\}$, and is also the case for $f(\gamma)$ and $f(\gamma')$ because each of these paths has a preimage path lying entirely in $Q_{0,0}$ or $Q_{0,2}$, and hence in Ω .

Therefore,

$$C \subset f(\Delta) \subset A_{M(R)}(f),$$

so $A_{M(R)}(f)$ has a bounded complementary component and hence is a spider's web, by [34, Theorem 1.4]. This is a contradiction. Thus, the uncountably many Eremenko points in $A_R(f)$ with distinct Wiman–Valiron itineraries lie in pairwise disjoint unbounded connected F_σ subsets of $A_R(f)$.

We have now shown that if $A_R(f)$ is *never* a spider's web, then for each $r_0 \geq R_1(f)$ there exists $R \in (r_0, 2r_0)$ such that $A_R(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ sets, each of the form $\Gamma(z(j_n))$, where $z(j_n)$

is an Eremenko point and $\Gamma(j_n)$ is the component of $A_R(f)$ that contains $z(j_n)$. To complete the proof of Theorem 1.3, we deduce that there exists a dense set of values R in $(R(f), \infty)$ with this property.

Suppose that $[R', R'']$ is any non-empty subinterval of $(R(f), \infty)$. Then, by the definition of $R(f)$, we have $M^n(R') \rightarrow \infty$ as $n \rightarrow \infty$, so it follows from (3.11), with $c = \log R'' / \log R'$, that there exists $N \in \mathbb{N}$ such that

$$M^N(R'') = M^N((R')^c) \geq M^N(R')^c \geq 2M^N(R') \geq 2R_1(f).$$

Therefore the interval $(M^N(R'), M^N(R''))$ contains an interval of the form $(r_0, 2r_0)$, where $r_0 \geq R_1(f)$.

By the earlier part of the proof, we can find $R_0 \in (r_0, 2r_0)$ such that $A_{R_0}(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ subsets of $A_{R_0}(f)$, each of the form $\Gamma(z)$, where z is an Eremenko point. Now put $R = M^{-N}(R_0)$. Then $R \in (R', R'')$ and we claim that $A_R(f)$ contains uncountably many pairwise disjoint unbounded connected F_σ subsets. For if z is an Eremenko point in $A_{R_0}(f)$, and $\zeta \in f^{-N}(z)$, then $\zeta \in A_R(f)$ and we can use (2.1) to define $\Gamma(\zeta)$ as an unbounded connected F_σ subset of $A_R(f)$ that contains ζ . In this way we obtain uncountably many unbounded connected F_σ sets, since $f^N(\Gamma(\zeta)) \subset \Gamma(z)$, and the uncountably many sets $\Gamma(z)$ are pairwise disjoint. This completes the proof of Theorem 1.3.

6. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 uses the sequences (r_n) , (z_n) , (Q_n) and (A_{n+1}) , $n \geq 0$, and the radius $R > 0$, which were defined in Theorem 3.3. Also, we put

$$C_n = \{z : |z| = 2r_n\}, \quad n \geq 0,$$

which, for $n \geq 1$, is a subset of the boundary of the cut annulus A_n .

Recall that the set Q_n , $n \geq 0$, consists of five quadrilaterals

$$Q_{n,j}, \quad j = -2, \dots, 2,$$

arranged in anticlockwise order around the origin, each of which is mapped one-to-one and conformally by f onto the cut annulus A_{n+1} . We also put

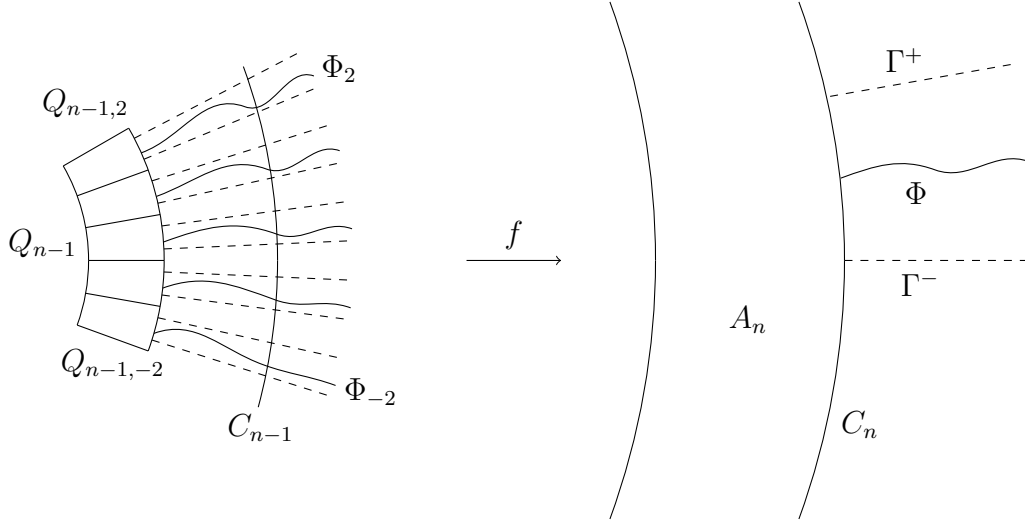
$$E_n = \partial Q_n \cap f^{-1}(C_{n+1}),$$

which is the outer edge of the quadrilateral Q_n .

The following concept is fundamental to our proof. An *escape channel* at level n , $n \geq 0$, is a triple $\Sigma = (\Gamma^-, \Phi, \Gamma^+)$, where Γ^-, Φ, Γ^+ are disjoint unbounded continua such that

- (a) Γ^-, Φ and Γ^+ all lie in $\{z : |z| \geq r_n\}$ and meet the circle C_n in closed sets that include points of the form $r_n e^{i\theta^-}$, $r_n e^{i\theta}$ and $r_n e^{i\theta^+}$, respectively, where $\theta^- < \theta < \theta^+ < \theta^- + 2\pi$;
- (b) $\Gamma^- \cup \Gamma^+ \subset I(f)^c$ and $\Phi \subset A_{M^n(R)}(f)$.

The *interior* of an n -th level escape channel $\Sigma = (\Gamma^-, \Phi, \Gamma^+)$ is the complementary component of $\Gamma^- \cup C_n \cup \Gamma^+$ that contains $\Phi \setminus C_n$. Two n -th level escape channels are called *disjoint* if their interiors are disjoint. For any n -th level escape channel Σ , the boundary of the interior of Σ contains a unique maximal closed subarc of C_n called the *entry* of Σ . If the entry of one n -th level escape

FIGURE 3. Preimages of an n -th level escape channel

channel Σ is a subset of the entry of another n -th level escape channel Σ' , then we write $\Sigma \prec \Sigma'$.

The next lemma shows that any n -th level escape channel can be pulled back under f and truncated to produce several disjoint $(n-1)$ -th level escape channels.

Lemma 6.1. *Let $(\Gamma^-, \Phi, \Gamma^+)$ be an n -th level escape channel, where $n \geq 1$. Then there exist four disjoint $(n-1)$ -th level escape channels,*

$$(\Gamma_k^-, \Phi_k, \Gamma_k^+), \quad k = 0, \dots, 3,$$

such that

$$(6.1) \quad f(\Gamma_k^-) \subset \Gamma^-, \quad f(\Phi_k) \subset \Phi, \quad f(\Gamma_k^+) \subset \Gamma^+, \quad \text{for } k = 0, \dots, 3.$$

Proof. Since f maps each quadrilateral $Q_{n-1,j}$, $j = -2, \dots, 2$, univalently onto the cut annulus A_n , we deduce that there are at least four distinct triples of preimage components of $(\Gamma^-, \Phi, \Gamma^+)$ that meet E_{n-1} in disjoint compact sets, which lie in order anticlockwise along E_{n-1} ; see Figure 3. All these preimage components are unbounded, by Lemma 2.1 (b); see also [6, Lemma 3.2]. Moreover, none of these preimage components has any points in the interior of Q_{n-1} . The preimage components of Φ are all in $A_{M^{n-1}(R)}(f)$ and the preimage components of Γ^- and Γ^+ are all in $I(f)^c$.

We choose four of these triples, which consist of twelve unbounded continua, all pairwise disjoint, and for each of these unbounded continua we take an unbounded closed connected subset of the intersection of the set with $\{z : |z| \geq 2r_{n-1}\}$ that meets C_{n-1} . This is possible by Corollary 2.2. These twelve subsets lie in the same order (around C_{n-1}) as do the twelve unbounded continua comprising the four triples (along E_{n-1}). Hence these twelve subsets form four disjoint escape channels at level $n-1$, which can be denoted by $(\Gamma_k^-, \Phi_k, \Gamma_k^+)$, $k = 0, \dots, 3$, and with this notation it follows by the construction that (6.1) holds. \square

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let Γ be an unbounded continuum lying entirely in $I(f)^c$. Without loss of generality we may suppose that $0 \in \Gamma$, by conjugating f with a translation if necessary. Then, for $n \geq 0$, we let $\Gamma_0(n, n)$ denote an unbounded component of $\Gamma \cap \{z : |z| \geq 2r_n\}$ that meets C_n , which is possible by Corollary 2.2. The reason for the notation $\Gamma_0(n, n)$ will become clear shortly.

We also introduce a single component, Φ_0 say, of $A_R(f)$ which contains an Eremenko point in the quadrilateral $\overline{Q_{0,0}}$, as constructed in Theorem 3.3. For $n \geq 1$, the set $f^n(\Phi_0)$ is contained in a component, Φ_n say, of $A_{M^n(R)}(f)$. Then $\Phi_n \cap Q_n \neq \emptyset$, by Theorem 3.3 (a), so we have $\Phi_n \cap C_n \neq \emptyset$. Now let $\Phi_0(n, n)$ denote an unbounded component of $\Phi_n \cap \{z : |z| \geq 2r_n\}$ that meets C_n , which is possible by Corollary 2.2 again.

The triple $(\Gamma_0(n, n), \Phi_0(n, n), \Gamma_0(n, n))$ can be thought of as a degenerate escape channel at level n , for which the two unbounded continua in $I(f)^c$ are identical. The proof of Lemma 6.1 can readily be adapted to show that there are four disjoint $(n-1)$ -th level escape channels,

$$(\Gamma_k^-(n-1, n), \Phi_k(n-1, n), \Gamma_k^+(n-1, n)), \quad k = 0, \dots, 3,$$

such that

$$(6.2) \quad f(\Gamma_k^\pm(n-1, n)) \subset \Gamma_0(n, n), \quad f(\Phi_k(n-1, n)) \subset \Phi_0(n, n), \quad \text{for } k = 0, \dots, 3.$$

We can now choose two of these four $(n-1)$ -th level escape channels with the additional property that the interior of neither of these escape channels meets Γ . This is possible since $\Gamma \subset I(f)^c$ and $\Phi_k(n-1, n) \subset A_{M^{n-1}(R)}(f)$ for $k = 0, \dots, 3$. We then relabel these two chosen escape channels as

$$(\Gamma_k^-(n-1, n), \Phi_k(n-1, n), \Gamma_k^+(n-1, n)), \quad k = 0, 1,$$

and note that (6.2) remains true.

We now apply Lemma 6.1 in this way repeatedly to produce, for all $n \geq 1$ and $0 \leq m < n$, a set of 2^{n-m} escape channels at level m , denoted by

$$\Sigma_k(m, n) = (\Gamma_k^-(m, n), \Phi_k(m, n), \Gamma_k^+(m, n)), \quad k = 0, \dots, 2^{n-m} - 1,$$

such that, for $k = 0, \dots, 2^{n-m} - 1$,

$$f(\Gamma_k^\pm(m, n)) \subset \Gamma_{[k/2]}^\pm(m+1, n), \quad f(\Phi_k(m, n)) \subset \Phi_{[k/2]}(m+1, n),$$

and, in addition,

$$\Sigma_k(m, n) \prec \Sigma_{[k/2]}(m, n-1), \quad \text{for } k = 0, \dots, 2^{n-m} - 1.$$

Hence we have, for $m = 0$, by induction,

$$(6.3) \quad \Gamma_k^\pm(0, n) \subset I(f)^c, \quad \Phi_k(0, n) \subset A_R(f), \quad \text{for } k = 0, \dots, 2^n - 1,$$

and

$$(6.4) \quad \Sigma_k(0, n) \prec \Sigma_{[k/2]}(0, n-1), \quad \text{for } k = 0, \dots, 2^n - 1,$$

For $n \geq 1$ and $k = 0, \dots, 2^n - 1$, we let $\sigma_k(0, n)$ denote the entry to the channel $\Sigma_k(0, n)$. Then each $\sigma_k(0, n)$ is a closed arc of C_0 which has endpoints in $\Gamma_k^\pm(0, n) \subset I(f)^c$ and contains a point of $\Phi_k(0, n) \subset A_R(f)$, by (6.3). Also,

$$(6.5) \quad \sigma_k(0, n) \subset \sigma_{[k/2]}(0, n-1), \quad \text{for } k = 0, \dots, 2^n - 1,$$

by (6.4). Now let

$$S_n = \bigcup_{k=0}^{2^n-1} \sigma_k(0, n), \quad n = 1, 2, \dots$$

Then (S_n) is a nested sequence of compact subsets of C_0 , whose intersection S has, by (6.5), uncountably many components, and all but at most countably many of these must be singletons.

Let $\{\zeta\}$ be a singleton component of S . Then we deduce that there is a sequence of points $\zeta_n \in C_0$ and integers $k(n)$, for $n \geq 1$, such that

$$(6.6) \quad \zeta_n \in \sigma_{k(n)}(0, n) \cap \Phi_{k(n)}(0, n) \text{ for } n \geq 1, \text{ and } \{\zeta\} = \bigcap_{n=1}^{\infty} \sigma_{k(n)}(0, n).$$

In particular, $\zeta_n \rightarrow \zeta$ as $n \rightarrow \infty$.

Then $\hat{\Phi}_{k(n)}(0, n) = \Phi_{k(n)}(0, n) \cup \{\infty\}$, $n \geq 0$, forms a sequence of continua in $\hat{\mathbb{C}}$, so we may assume, by taking a subsequence if necessary, that $\hat{\Phi}_{k(n)}(0, n)$ converges with respect to the Hausdorff metric on $\hat{\mathbb{C}}$ to a continuum in $\hat{\mathbb{C}}$ containing ∞ and ζ ; see [14, pages 37–39]. Since $A_R(f)$ is closed, the part of this limiting continuum in \mathbb{C} is contained in $A_R(f)$. Hence ζ lies in an unbounded closed connected subset of $A_R(f)$, which we denote by Φ_ζ . We denote by I_ζ the component of $I(f) \cap \{z : |z| \geq 2r_0\}$ that contains Φ_ζ .

Suppose now that ζ and ζ' are distinct singleton components of S . Then we claim that I_ζ and $I_{\zeta'}$ are disjoint. Indeed, it follows by (6.6) that there are unbounded continua in $I(f)^c$, which lie in $\{z : |z| \geq 2r_0\}$ and which meet C_0 at points that lie on either side of ζ and as close as we like to ζ . This proves our claim.

To summarise what we have proved, the set $I(f) \cap \{z : |z| \geq 2r_0\}$ has uncountably many components I_ζ , $\zeta \in S$, each of which contains an unbounded continuum Φ_ζ such that $\zeta \in \Phi_\zeta \subset A_R(f)$. Moreover, for any two distinct singleton components of S , $\{\zeta\}$ and $\{\zeta'\}$ say, there are unbounded continua in $I(f)^c$, which lie in $\{z : |z| \geq 2r_0\}$ and separate I_ζ from $I_{\zeta'}$.

Now let D be any open disc that meets $J(f)$. Then, by the blowing up property of $J(f)$, there exists $N \in \mathbb{N}$ such that

$$D_n = \widetilde{f^n(D)} \supset \{z : |z| \leq 2r_0\}, \text{ for } n \geq N.$$

Here the notation \widetilde{U} denotes the union of the set U with all its bounded complementary components. By what we proved above, the set $\mathbb{C} \setminus D_N$ contains uncountably many components of $I(f) \setminus D_N$, each meeting ∂D_N and containing an unbounded continuum in $A_R(f)$, and each pair of these components is separated in $I(f) \setminus D_N$ by an unbounded continuum in $I(f)^c$.

Since $\partial D_N \subset \partial f^N(D) \subset f^N(\partial D)$, we deduce that there is an arc α of ∂D such that $f^N(\alpha)$ is an arc of ∂D_N that contains points of uncountably many components of $I(f) \setminus D_N$ each containing an unbounded continuum in $A_R(f)$, and each pair of which is separated in $\mathbb{C} \setminus D_N$ by an unbounded continuum in $I(f)^c$ that meets $f^N(\alpha)$.

Hence, by another application of Lemma 2.1, there are uncountably many components of $I(f) \setminus D$ each containing an unbounded continuum in

$$f^{-N}(A_R(f)) \subset A(f) \subset I(f),$$

and each pair of which is separated in $\mathbb{C} \setminus D$ by an unbounded continuum in $I(f)^c$ that meets the arc α . This completes the proof of Theorem 1.2. \square

Remark The analogous result to Theorem 1.2 concerning the set $A(f)$ has a proof that is identical to the proof of Theorem 1.2, with $I(f)$ replaced by $A(f)$ throughout.

7. PROPERTIES OF MULTIPLY CONNECTED WANDERING DOMAINS

In this section, we recall some known properties of multiply connected wandering domains, which are needed to prove Theorems 1.5 and 1.6. In the first result, we give some basic properties, including the result of Baker that, for transcendental entire functions, multiply connected wandering domains are the only multiply connected Fatou components.

Lemma 7.1. *Let f be a transcendental entire function, let U be a multiply connected Fatou component of f , let $U_n = f^n(U)$ for $n \in \mathbb{N}_0$, and suppose that $R > R(f)$. Then U is a bounded wandering domain and, more precisely,*

- (a) *each U_n , $n \in \mathbb{N}$, is a bounded Fatou component of f ;*
- (b) *U_{n+1} surrounds U_n for sufficiently large n , and $\text{dist}(\partial U_n, 0) \rightarrow \infty$ as $n \rightarrow \infty$;*
- (c) *$\overline{U_n} \subset A_R(f)$, for sufficiently large n , and $A_R(f)$ is a spider's web;*
- (d) *all the components of $J(f)$ and hence of $A_R(f) \cap J(f)$ are bounded;*
- (e) *each component of ∂U_n , for sufficiently large n , is contained in a distinct component of $A_R(f) \cap J(f)$;*
- (f) *f has no exceptional points, that is, no points with a finite backward orbit.*

See [1, Theorem 3.1] for properties (a) and (b), and [34, Theorem 4.4, Corollary 6.1 and Theorem 1.3] for properties (c), (d) and (e). Property (f) holds because if f has an exceptional point α , then $f(z) = \alpha + (z - \alpha)^m \exp(g(z))$, where $m \geq 0$ and g is entire [5, Section 2.2], and it follows that α is an asymptotic value of f , which is impossible by property (b).

In order to define the notion of inner connectivity, which is used in the statement of Theorem 1.6, we need the following result [7, Theorem 1.3]. This strengthens an earlier result of Zheng [38] showing that multiply connected wandering domains contain large annuli.

Lemma 7.2. *Let f be a transcendental entire function with a multiply connected wandering domain U , let $z_0 \in U$ and put $r_n = |f^n(z_0)|$ and $U_n = f^n(U)$ for $n \in \mathbb{N}_0$. Then there exist $\alpha > 0$ and sequences (a_n) and (b_n) with*

$$0 < a_n < 1 - \alpha < 1 + \alpha < b_n, \quad \text{for } n \in \mathbb{N}_0,$$

such that, for sufficiently large $n \in \mathbb{N}$,

$$B_n = A(r_n^{a_n}, r_n^{b_n}) \subset U_n.$$

Moreover, for every compact subset C of U , we have $f^n(C) \subset B_n$ for $n \geq N(C)$.

In view of this last property we often describe the large annuli B_n as ‘absorbing’. We then define the *inner connectivity* of U_n to be the connectivity of the domain $U_n \cap \{z : |z| < r_n\}$ and the *outer connectivity* of U_n to be the connectivity of the

domain $U_n \cap \{z : |z| > r_n\}$. We also define the *outer boundary component* of a bounded domain U to be the boundary of the unbounded component of $\mathbb{C} \setminus U$, denoted by $\partial_{\text{out}}U$, and the *inner boundary component* of U to be the boundary of the component of $\mathbb{C} \setminus U$ that contains 0, if there is one, denoted by $\partial_{\text{inn}}U$.

The inner connectivity of a multiply connected wandering domain can behave in one of two ways, given by the following lemma; see [7, Theorem 8.1(b)].

Lemma 7.3. *Let f be a transcendental entire function, let U be a multiply connected wandering domain of f , and let $U_n = f^n(U)$ for $n \in \mathbb{N}_0$. Then there exists $N \in \mathbb{N}$ such that exactly one of the following holds:*

- (a) U_n has infinite inner connectivity for all $n \geq N$;
- (b) U_n has finite inner connectivity for all $n \geq N$, which decreases with n , eventually reaching the value 2.

Remark It is clear from this lemma that the concept of *eventual inner connectivity* of a multiply connected wandering domain (see [7]) is well defined, and that the eventual inner connectivity can take the values infinity or 2.

To prove Theorems 1.5 and 1.6, we use another property of multiply connected wandering domains, proved as part of [35, Lemma 3.2].

Lemma 7.4. *Let f be a transcendental entire function, let U be a multiply connected wandering domain of f , and let $U_n = f^n(U)$ for $n \in \mathbb{N}_0$. Then there exists $N \in \mathbb{N}$ and a sequence of annuli $B'_n = A(r'_n, r''_n) \subset U_n$, for $n \geq N$, such that*

$$(7.1) \quad f(B'_n) \subset B'_{n+1}, \quad \text{for } n \geq N.$$

In particular, if $R \in (r'_N, r''_N)$, then

$$(7.2) \quad \{z : |z| = M^{n-N}(R)\} \subset U_n, \quad \text{for } n \geq N.$$

The final result in this section describes the three possible types of complementary components that a multiply connected wandering domain can have. We discuss the possible existence of these types of components in the next section.

Lemma 7.5. *Let f be a transcendental entire function with a multiply connected wandering domain U and let K be a bounded complementary component of U .*

- (a) *For all $n \in \mathbb{N}$, the set $f^n(K)$ is a bounded complementary component of $f^n(U)$.*
- (b) *The component K is of one of the following types:*
 1. *the interior of K meets $J(f)$ and, for sufficiently large $n \in \mathbb{N}$, the set $f^n(K)$ is the complementary component of $f^n(U)$ that contains 0;*
 2. *the interior of K is a union of Fatou components;*
 3. *K has empty interior.*
- (c) *If K is of type 2 or type 3, then every point of ∂K is the limit of points lying in distinct type 1 complementary components.*

Proof. The result of part (a) may be well known, but we include a proof for completeness; see also [20, Theorem 3.1(ii)] for the result of part (a) in a more general setting. It is sufficient to consider the case $n = 1$.

Since $f(K)$ is connected there is a unique complementary component, L say, of $V = f(U)$ such that $f(K) \subset L$. We show that $f(K) = L$.

Let $V_n, n \in \mathbb{N}$, be a smooth exhaustion of V ; that is, V_n are smooth domains such that $\overline{V_n} \subset V_{n+1}$, for $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} V_n = V$. Then L lies in a unique component, H_n say, of $\mathbb{C} \setminus \overline{V_n}$, for each $n \in \mathbb{N}$. Each H_n is a Jordan domain with its boundary in V , $\overline{H_{n+1}} \subset H_n$, for $n \in \mathbb{N}$, and $\bigcap_{n \in \mathbb{N}} \overline{H_n} = L$, since (V_n) is an exhaustion of V .

Now let $G_n, n = 1, 2, \dots$, denote the component of $f^{-1}(H_n)$ that contains K . Then $\partial G_n \subset U$ and $f : G_n \rightarrow H_n$ is a proper map, so $\overline{G_{n+1}} \subset G_n$, for $n \in \mathbb{N}$. Then $K' = \bigcap_{n \in \mathbb{N}} \overline{G_n}$ is a compact connected set such that $f(K') = L$. Indeed, $f(\overline{G_n}) = \overline{H_n}$, for $n \in \mathbb{N}$, so the inclusion

$$f\left(\bigcap_{n \in \mathbb{N}} \overline{G_n}\right) \subset \bigcap_{n \in \mathbb{N}} \overline{H_n}$$

is clear, and the reverse inclusion also holds since if $w \in \bigcap_{n \in \mathbb{N}} \overline{H_n}$, then there exists $z_n \in \overline{G_n}$ such that $f(z_n) = w$, for all $n \in \mathbb{N}$, and hence $f(z) = w$ for some $z \in \bigcap_{n \in \mathbb{N}} \overline{G_n}$, by compactness.

We now show that $K' = K$. Clearly, $K \subset K'$ and also $K' \subset \mathbb{C} \setminus U$, since $f(K') = L \subset \mathbb{C} \setminus V$, so $K' = K$ and hence $f(K) = L$, as required.

To prove part (b), suppose that the interior of K meets $J(f)$. Since f has no exceptional values, the backward orbit of 0 accumulates at every point of $J(f)$, so we deduce that $\text{int } K$ must contain a point z such that for some $n \in \mathbb{N}$ we have $f^n(z) = 0$. It follows by part (a) that $f^n(K)$ is the complementary component of $f^n(U)$ that contains 0.

Since $\partial K \subset J(f)$, part (c) also follows immediately from the fact that the backward orbit of 0 accumulates at every point of $J(f)$. \square

8. COMPLEMENTARY COMPONENTS OF MULTIPLY CONNECTED WANDERING DOMAINS

In this section we prove the following result, which arose from discussions with Markus Baumgartner and Walter Bergweiler. We use Theorem 8.1 in the proof of Theorem 1.6, and it also has considerable interest in its own right. For example, it shows that in some cases a multiply connected wandering domain can have uncountably many complementary components of type 3, that is, ones with no interior. It was not previously known whether such complementary components could exist.

Theorem 8.1. *Let f be a transcendental entire function with a multiply connected wandering domain U , let N be so large that the inner and outer connectivity of U_N are defined.*

- (a) *If U_N has infinite inner connectivity, then*
 - (i) *U_N has uncountably many complementary components that accumulate at the inner boundary component of U_N ;*
 - (ii) *U_N has uncountably many complementary components with no interior (type 3), as has U ;*

- (iii) *the outer connectivity of U_N is either 2 or uncountable.*
- (b) *If U_N has finite inner connectivity, then the outer connectivity of U_N is finite or countable, and the complementary components accumulate nowhere in $\overline{U_N}$ except possibly at the outer boundary component of U_N .*

As far as we know, it is an open question whether a multiply connected wandering domain can have type 2 complementary components and also whether it can have complementary components that are *singleton* sets.

We have the following corollary of Theorem 8.1 (b), together with Lemma 7.5 (c) and Lemma 7.3 (a), which was also given in Baumgartner's PhD thesis [4, Theorem 3.1.25]; see the remark after Lemma 7.3 for the meaning of 'eventual inner connectivity'.

Corollary 8.2. *Let f be a transcendental entire function with a multiply connected wandering domain U . Then U has eventual inner connectivity 2 if and only if all the complementary components of U are of type 1.*

Examples of transcendental entire functions with multiply connected wandering domains having either infinite inner connectivity or finite inner connectivity were given in [9, Remarks following Theorem 1.3]. Other examples with finite inner connectivity were given in [10] and in [18].

Proof of Theorem 8.1. (a) Put $n_0 = N$. Since U_{n_0} has infinite inner connectivity and $J(f)$ is closed, we deduce from Lemma 2.1 (b) that there exists a Jordan curve γ_0 in U_{n_0} which surrounds at least one component of $U_{n_0}^c$ and does not surround 0. Then, by Lemma 7.2, together with the argument principle, there exists $n_1 \in \mathbb{N}$ such that $f^{n_1-n_0}(\gamma_0)$, lies in the absorbing annulus B_{n_1} and winds at least once round 0. By Lemma 7.3, we can take disjoint Jordan curves $\gamma_{n_1,1}$ and $\gamma_{n_1,2}$ in U_{n_1} , with disjoint interiors, which each surround at least one component of $U_{n_1}^c$, do not surround 0, and lie in the bounded component of $B_{n_1}^c$, and then repeat the process above for each of $\gamma_{n_1,1}$ and $\gamma_{n_1,2}$.

Continuing in this way, we can construct a strictly increasing sequence of positive integers (n_m) and, for $m \in \mathbb{N}$, disjoint Jordan curves $\gamma_{n_m,1}$ and $\gamma_{n_m,2}$, with disjoint interiors, which each surround at least one component of $U_{n_m}^c$, do not surround 0, and lie in the bounded component of $B_{n_m}^c$, such that

$$f^{n_{m+1}-n_m}(\gamma_{n_m,j}) \text{ surrounds } \gamma_{n_{m+1},k}, \text{ for } m \in \mathbb{N}, j, k = 1, 2.$$

We now show that there exist points in $J(f)$ whose images under f^{n_m} lie in the interior of any specified choice of the Jordan curves γ_{n_m,j_m} , for $m \in \mathbb{N}$, $j_m \in \{1, 2\}$. To prove this we consider the sequence of compact sets (E_n) , defined as follows:

- for $n = n_m, m \geq 0$, we take E_n to be the union of γ_{n_m,j_m} and $\text{int } \gamma_{n_m,j_m}$;
- for $n_m < n < n_{m+1}, m \geq 0$, we take E_n to be $f^{n-n_m}(E_{n_m})$.

It is clear that the sequence $(E_{n_0+k}), k \geq 0$, satisfies all the hypotheses of Lemma 2.4, so there exists $\zeta \in J(f)$ such that $f^k(\zeta) \in E_{n_0+k}$ for $k \geq 0$ and, in particular,

$$(8.1) \quad f^{n_m-n_0}(\zeta) \text{ is surrounded by } \gamma_{n_m,j_m}, \text{ for } m \in \mathbb{N}.$$

Since we have two choices of the Jordan curve at each stage (after the first), this gives rise to uncountably many points of $J(f)$, each of which has the property that its images lie in the interiors of the specified Jordan curves. Each such point, ζ say, must be contained in a complementary component, K_ζ say, of U_{n_0} , and we claim that if (8.1) holds, then

$$(8.2) \quad f^{n_m - n_0}(K_\zeta) \text{ is surrounded by } \gamma_{n_m, j_m}, \text{ for } m \in \mathbb{N},$$

from which it follows that all such complementary components K_ζ , arising from points with different ‘itineraries’ (j_m), are distinct. Hence $U_{n_0} = U_N$ has uncountably many complementary components, as required.

To deduce (8.2) from (8.1), we note that, for $m \in \mathbb{N}$, each complementary component of U_{n_0} must map under $f^{n_m - n_0}$ onto a complementary component of U_{n_m} , by Lemma 7.5. It follows, in particular, that the complementary component K_ζ must map under $f^{n_m - n_0}$ to the complementary component of U_{n_m} that contains $f^{n_m}(\zeta)$, so this complementary component of U_{n_m} must be surrounded by γ_{n_m, j_m} . This proves (8.2).

These complementary components of U_{n_0} must accumulate at the inner boundary component of U_{n_0} for otherwise we could find a Jordan curve $\gamma \subset U_{n_0}$ that surrounds the inner boundary component of U_{n_0} but no other boundary components. For n sufficiently large $f^n(\gamma)$ must lie in B_{n_0+n} and wind at least once round 0, which contradicts the fact that U_{n_0+n} has infinite inner connectivity. This proves part (a)(i). It follows that U_N has uncountably many complementary components with no interior, since it can have only countably many with interior. To obtain the same result for U , we can apply a similar argument to that in the above proof, but start with a Jordan curve in U that surrounds at least one complementary component of U . This proves part (a)(ii).

To prove part (a)(iii), we observe that if a complementary component, K say, of U_{n_0} exists outside B_{n_0} and γ is a Jordan curve in U_{n_0} that surrounds K , then for n sufficiently large $f^n(\gamma)$ must lie in B_{n_0+n} and wind at least once round 0, and hence γ must surround uncountably many complementary components of U_{n_0} .

To prove part (b), we suppose that the inner connectivity of U_N is finite. Let γ be a Jordan curve in U_N that surrounds at least one boundary component of U_N . Then there exists $n \in \mathbb{N}$ such that $f^n(\gamma)$ lies in B_{N+n} and winds at least once round 0. Then $f^n(\gamma)$ winds round at most finitely many components of U_{N+n}^c , so γ surrounds at most finitely many components of U_N^c . It follows that U_N has at most countably many complementary components, so the outer connectivity of U_N is at most countable, and also that the complementary components of U_N do not accumulate at any point of $\overline{U_N}$ except possibly at the outer boundary component of U_N . \square

9. PROOFS OF THEOREMS 1.5 AND 1.6

In this section we prove our results about the components of the sets $I(f) \cap J(f)$, $A(f) \cap J(f)$ and $A_R(f) \cap J(f)$ in the case when f has a multiply connected wandering domain.

We begin by proving Theorem 1.5. Here we use a variation of an argument we introduced in [32] and [35].

Proof of Theorem 1.5. Theorem 1.5 states that for any transcendental entire function f with a multiply connected wandering domain, the set $I(f) \cap J(f)$ has uncountably many components. Given such a function f , let B'_n , $n \geq 0$, be the open annuli given by Lemma 7.4 and for each $n \geq 0$ let E_n denote the closed annulus lying precisely between B'_n and B'_{n+1} . Then, any component, Γ say, of $I(f) \cap J(f)$ must satisfy

$$f^j(\Gamma) \subset E_{n_j}, \text{ for } j \geq 0,$$

for some sequence (n_j) , since the annuli $B'_n \subset F(f)$ for $n \geq 0$.

We show that uncountably many components of $I(f) \cap J(f)$ arise in this way. By (7.1),

$$\partial f(E_n) \subset f(\partial E_n) \subset \overline{B'_{n+1}} \cup \overline{B'_{n+2}}, \text{ for } n \geq 0,$$

so

$$(9.1) \quad f(E_n) \supset E_{n+1}, \text{ for } n \geq 0.$$

Also, since the annuli B'_n lie in distinct Fatou components of f , we deduce that

$$(9.2) \quad E_n \cap J(f) \neq \emptyset, \text{ for } n \geq 0.$$

Next we put

$$E'_n = B'_n \cup E_n \cup B'_{n+1}, \text{ for } n \geq 0,$$

and let F_n denote the bounded component of $\mathbb{C} \setminus E'_n$. Then it follows from (7.1) that, for each $n \geq 0$, we have exactly one of the following possibilities:

$$(9.3) \quad f(E'_n) \subset E'_{n+1},$$

or

$$(9.4) \quad f(E'_n) \supset F_{n+1}, \text{ so } f(E_n) \supset F_{n+1} \supset E_n.$$

If (9.3) holds for all $n \geq N$, say, then each E'_n , $n \geq N$, is contained in the Fatou set of f , by Montel's theorem, and this contradicts the fact that each E_n and hence each E'_n meets $J(f)$. Thus there is a strictly increasing sequence $n_j \geq 0$ such that (9.4) holds for $n = n_j$, $j \in \mathbb{N}$, so

$$(9.5) \quad f(E_{n_j}) \supset E_{n_j}, \text{ for } j \in \mathbb{N}.$$

We now observe that there are uncountably many increasing sequences s of non-negative integers, each of which includes all the non-negative integers and some repetitions of the integers n_j , $j \in \mathbb{N}$. For each of these sequences, the properties (9.1), (9.2) and (9.5) allow us to apply Lemma 2.4 to give a point in $J(f)$ whose orbit passes through the annuli E_n in a manner determined by the sequence s . For two such distinct sequences s , we obtain (since the annuli B'_n all lie in $F(f)$) two distinct components of $I(f) \cap J(f)$. (Note that we can obtain a component with a prescribed rate of escape by specifying an appropriate sequence s .) Since there are uncountably many distinct such sequences s , there are uncountably many distinct components of $I(f) \cap J(f)$, as required. \square

Finally we prove Theorem 1.6.

Proof of Theorem 1.6. Part (a) states that if U is a multiply connected wandering domain of a transcendental entire function f with infinite inner connectivity and $R > R(f)$, then $A_R(f) \cap J(f)$ and $A(f) \cap J(f)$ have uncountably many components. By Lemma 7.1 (c) we can take N so large that $\overline{U_N} = \overline{f^N(U)} \subset A_R(f)$. By Theorem 8.1 (a)(i), we know that U_N has uncountably many complementary components. The boundaries of these complementary components are subsets of $A_R(f) \cap J(f)$ and $A(f) \cap J(f)$, so the result follows.

Part (b) states that there exists a transcendental entire function f with a multiply connected wandering domain and $R > R(f)$ such that $A_R(f) \cap J(f)$ and $A(f) \cap J(f)$ each have only countably many components. This property holds for a remarkable example constructed by Bishop of a transcendental entire function f whose Julia set has dimension 1; see [10, Theorem 1.3]. For the reader's convenience, we outline the proof of this property of the components of $A_R(f) \cap J(f)$ and $A(f) \cap J(f)$.

Bishop's function has a multiply connected wandering domain U whose forward orbit $U_n = f^n(U)$ has the following topological properties. For $n \geq 0$,

- the boundary components of U_n are all Jordan curves;
- the inner boundary component of U_{n+1} is identical to the outer boundary component of U_n ;
- the outer connectivity of U_n is countably infinite and the inner connectivity is 2.

By Lemma 7.2, we can also assume that there exists $R > R(f)$ such that

$$(9.6) \quad \{z : |z| = M^n(R)\} \subset U_n, \text{ for } n \geq 0.$$

In fact Bishop's proof in [10] gives the property (9.6) as a part of the construction.

It turns out that for this function f any point $\zeta \in A(f) \cap J(f)$ must lie in one of the countably many boundary components of one of the domains U_n or a pre-image of such a boundary component. If not, then it follows, by the properties of the wandering domain given above, that, for all $n \geq 0$, the point $f^n(\zeta)$ must lie in the interior of a type 1 complementary component of $U_{k(n)}$, for some integer $k(n)$. By Lemma 7.5 (b), any such complementary component of $U_{k(n)}$ must map, for some $m \in \mathbb{N}$, to the complementary component of $f^m(U_{k(n)}) = U_{k(n)+m}$ that contains 0, so $f^{m+n}(\zeta)$ belongs to a type 1 complementary component of U_ℓ for some $\ell \leq k(n) + m - 1$. Therefore, there exists a sequence of positive integers (n_j) such that

$$k(n_j) \leq k(n_{j-1}) + n_j - n_{j-1} - 1, \text{ for } j \in \mathbb{N},$$

and hence

$$k(n_j) - n_j \leq d - j, \text{ for } j \in \mathbb{N},$$

where $d = k(n_0) - n_0$. Thus

$$f^{n_j}(\zeta) \in \tilde{U}_{d+n_j-j}, \text{ for } j \in \mathbb{N}.$$

Recall that the notation \tilde{V} denotes the union of V with its bounded complementary components. Therefore, by (9.6),

$$|f^{n_j}(\zeta)| \leq M^{d+n_j-j+1}(R), \text{ for } j \in \mathbb{N},$$

which implies that $\zeta \notin A(f)$, a contradiction. Hence there are only countably many components of $A(f) \cap J(f)$, and similarly only countably many components of $A_R(f) \cap J(f)$. This completes the proof of Theorem 1.6. \square

We remark that a large class of transcendental entire functions with topological properties similar to Bishop's example was constructed by Baumgartner [4].

10. OPEN QUESTIONS

In this final section we discuss several interesting questions, which arise in connection with our new results.

Question 10.1. *Let f be a transcendental entire function. For each of the sets $I(f)$, $A(f)$ and $A_R(f)$, where $R > R(f)$, is it the case that the set is either connected or it has uncountably many components?*

Theorem 1.2 gives a partial answer to this question for $I(f)$, since it states that if $I(f)^c$ contains an unbounded continuum, in particular if $I(f)$ is disconnected, then the set $I(f) \setminus D$, where D is any open disc that meets $J(f)$, has uncountably many unbounded components, and there is a similar partial result for $A(f)$. These results raise the following question about $I(f)$, and there is a similar question about $A(f)$.

Question 10.2. *Does there exist a transcendental entire function f such that $I(f)$ is connected and $I(f)^c$ contains an unbounded continuum?*

As noted in the introduction, the function $f(z) = e^z$ has the property that $I(f)$ is connected and there is an unbounded connected set in the complement of $I(f)$; see [23, Example 2]. We do not know, however, whether such an unbounded connected set can be taken to be closed for this function. Note that in our proof of Theorem 1.2 we make strong use of the fact that the unbounded connected set Γ in $I(f)^c$ is closed.

Theorem 1.3 is a step towards answering Question 10.1 for $A_R(f)$ but it leaves a number of questions open, which we now collect together.

- Question 10.3.**
- (a) *In Theorem 1.3, can we replace the ‘unbounded connected F_σ sets’ in $A_R(f)$ by ‘unbounded continua’ in $A_R(f)$ or even better by ‘components’ of $A_R(f)$?*
 - (b) *In Theorem 1.3, can we replace the ‘dense set’ of values of $R > R(f)$ by ‘all’ $R > R(f)$?*
 - (c) *Does there exist a transcendental entire function f such that the set $A_R(f)$ is connected for some $R > 0$ but $A_R(f)$ is not a spider’s web?*

We conclude by stating two questions about the complementary components of multiply connected wandering domains. First recall that in Lemma 7.5 (b) we described three types of complementary components of multiply connected wandering domains. It is well known that there are many examples of type 1 and we showed in Theorem 8.1 (a) (ii) that type 3 complementary components, that is, ones with no interior, occur whenever a multiply connected wandering domain has infinite inner connectivity.

Question 10.4. *Let f be a transcendental entire function and let U be a multiply connected wandering domain of f . Is it possible for U to have complementary components of type 2, that is, ones with interior that is a union of Fatou components?*

Corollary 8.2 shows that if such type 2 components exist, then U must have infinite inner connectivity.

Question 10.5. *Let f be a transcendental entire function and let U be a multiply connected wandering domain of f with infinite inner connectivity. Is it possible, or indeed necessary, that U has (uncountably many) singleton complementary components?*

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SCHOOL OF MATHEMATICS AND STATISTICS, THE OPEN UNIVERSITY, WALTON HALL, MILTON KEYNES MK7 6AA, UK

E-mail address: phil.rippon@open.ac.uk

SCHOOL OF MATHEMATICS AND STATISTICS, THE OPEN UNIVERSITY, WALTON HALL, MILTON KEYNES MK7 6AA, UK

E-mail address: gwyneth.stallard@open.ac.uk